# Generalization of parity-time and partial parity-time symmetry

**Francisco M. Fernández**

INIFTA (UNLP, CCT La Plata-CONICET), Blvd. 113 y 64 S/N, Sucursal 4, Casilla de Correo 16, 1900 La Plata, Argentina

E-mail: `fernande@quimica.unlp.edu.ar`

**Abstract.** We show that parity-time and partial parity-time symmetries are particular cases of antiunitary symmetry. This point is illustrated by means of a recently discussed system of non-Hermitian coupled harmonic oscillators that also exhibits other types of antiunitary symmetries. We also show that a combination of group and perturbation theory is a useful tool for predicting broken antiunitary symmetry.

## 1. Introduction

In a recent series of papers Fernández and Garcia [1, 2] and Amore et al [3, 4] applied group theory to non-Hermitian coupled oscillators with the purpose of determining the conditions under which the well known parity-time (PT) symmetry [5] as well as its generalization space-time (ST) symmetry [6] are unbroken. One of the examples discussed by Fernández and Garcia [1] is a pair of harmonic oscillators coupled by an imaginary quadratic interaction $H_{2D} = p_x^2 + p_y^2 + \omega_x^2 x^2 + \omega_y^2 y^2 + iaxy$. They found that when the two oscillators are identical ($\omega_x^2 = \omega_y^2$) the eigenvalues $E_{mn}$ are real when $m = n$ and pairs of complex conjugate numbers when $m \neq n$. On the other hand, when $\omega_x^2 \neq \omega_y^2$ all the eigenvalues are real for certain combinations of the oscillator parameters $\omega_x^2$, $\omega_y^2$ and $a$. That is to say: in the latter case there is a sort of phase transition from real to complex eigenvalues.

More recently, Beygi et al [8] discussed a family of $N$ coupled oscillators that contain the example discussed by Fernández and Garcia when $N = 2$. These authors assumed that each oscillator interacts only with its nearest neighbors in a sort of linear chain; that is to say: the $i$-th oscillator interacts only with the $(i-1)$-th and $(i+1)$-th ones. They studied both quantum-mechanical as well as classical oscillators and arrived at exactly the same conclusions drawn earlier by Fernández and Garcia [1] for the former case. According to Beygi et al those non-Hermitian coupled harmonic oscillators exhibit partial PT symmetry which is a concept introduced somewhat earlier by Yang [9]. Both PT symmetry and partial PT symmetry are particular cases of the

ST symmetry introduced by Klainman and Cederbaun [6] and all of them are examples of the antiunitary symmetry discussed by Wigner [7] long time before. It is well known that if $U$ is a unitary transformation in configuration space and $T$ is the time reversal operation then $A_U = UT$ is an antiunitary operator [7]. Therefore, it seems more reasonable to speak of antiunitary symmetry as the most general concept. When $A_U$ leaves the non-Hermitian Hamiltonian operator $H$ invariant ($A_U H A_U^{-1} = H$) we say that $H$ exhibits antiunitary symmetry. In the particular case that $U$ is the parity operation $P$ then we are in the presence of PT symmetry. According to Yang [9] when the potential energy function $V(x, y)$ of a two-dimensional quantum-mechanical model satisfies $V(-x, y)^* = V(x, y)$ or $V(x, -y)^* = V(x, y)$ then the model satisfies partial PT symmetry (note that $V(-x, -y)^* = V(x, y)$ describes the usual PT symmetry). Obviously, it is a particular case of the antiunitary symmetry generated by the unitary operators $U_x : (x, y) \to (-x, y)$ or $U_y : (x, y) \to (x, -y)$, respectively. Note, for example, that the Hamiltonian $H_{2D}$ is invariant with respect to the antiunitary operators $A_x = U_x T$ and $A_y = U_y T$. In fact, Klainman and Cederbaum [6] chose $H = H_0 + i\lambda xy$ as a simple prototypical example of ST symmetry. The exactly-solvable $N$-dimensional harmonic oscillators chosen by Beygi et al [8] that exhibit partial PT symmetry are also particular examples of antiunitary symmetry.

By means of group theory and perturbation theory Amore et al [3] conjectured that of all the examples of antiunitary symmetry studied so far, PT symmetry appears to be the less likely to be broken. Their conclusion was based on the role of $U$ in the symmetry group $G_0$ for the unperturbed Hermitian oscillator $H_0$.

In a later communication Fernández and Garcia [11] discussed three PT-symmetric models with completely different spectra. They are of the form $H = H_0 + igz$, where $H_0$ is an Hermitian operator with a central-field potential $V(r)$. The first one is exactly solvable and exhibits a real spectrum for all values of $g$. On the other side, the second one exhibits complex eigenvalues for all values of $g$. Finally, with an intermediate behavior, the third one shows the well known phase transition typical of PT-symmetric quantum-

mechanical models [10]. The second model is most interesting because the search carried out by Fernández and Garcia [1, 2] and Amore et al [3, 4] failed to produce any PT-symmetric Hamiltonian with completely broken PT symmetry. In this case the phase transition takes place at the trivial Hermitian limit $g = 0$. The remarkable difference in the spectra of those non-Hermitian operators can be traced back to the symmetry of $H_0$. The higher this symmetry the more probable the appearance of complex eigenvalues [11].

One of the aims of this paper is to analyse the system of coupled oscillators proposed by Beygi et al [8] from the point of view of group theory and perturbation theory. In Sec. 2 we develop the concept of antiunitary symmetry in a way that generalizes and extends the original idea of Klainman and Cederbaum [6] and improves the point-group analysis carried out by Amore et al [3]. We also show that the combination of group theory and perturbation theory yields a fairly good idea about the kind of spectrum that one expects of a given non-Hermitian operator. In Sec. 3 we study the $N$-dimensional coupled harmonic oscillators of Beygi et al [8] from the point of view of point-group theory. In addition to finding the point group for these oscillators we show that their antiunitary symmetry is greater than the partial PT symmetry discussed so far. Finally, in Sec. 4 we summarize the main results of the paper and draw conclusions.

## 2. Antiunitary symmetry

Consider a Hamiltonian operator of the form

$$H(\lambda) = H_0 + \lambda H', \tag{1}$$

where $H_0$ is Hermitian and $H'$ is real and linear. Suppose that there is a group $G = \{U_1, U_2, \ldots, U_n\}$ of unitary operators ($U_i^\dagger = U_i^{-1}$) that leave both $H_0$ and $H'$ invariant:

$$U_i H_0 U_i^{-1} = H_0,\ U_i H' U_i^{-1} = H'. \tag{2}$$

This group of operators is commonly called the symmetry group for $H$ [12,13]. Suppose that there is a set of unitary operators $S_W = \{W_1, W_2, \ldots, W_m\}$ such that

$$W_i H_0 W_i^{-1} = H_0, \; W_i H' W_i^{-1} = -H'. \tag{3}$$

Since $U_i W_j \in S_W$ and $W_i W_j \in G$ we conclude that $G_0 = G \cup S_W$ is a group of unitary operators that leave $H_0$ invariant. Obviously, $G_0$ is at least a subgroup of the symmetry group for $H_0$.

Let us now consider the set of antiunitary operators $S_A = \{A_1, A_2, \ldots, A_m\}$, where $A_i = W_i T = T W_i$ and $T$ is the time-reversal operator [7]. If $\lambda^* = -\lambda$ then $A_i H(\lambda) A_i^{-1} = H(-\lambda^*) = H(\lambda)$. Besides, since $U_i A_j \in S_A$ and $A_i A_j \in G$ we conclude that $G_A = G \cup S_A$ is a group of operators that leave $H(\lambda)$ invariant

$$K H(\lambda) K^{-1} = H(\lambda), \; K \in G_A, \; \lambda^* = -\lambda. \tag{4}$$

We call $G_A$ the antiunitary-symmetry group for $H$ as a generalization of the concept introduced by Klainman and Cederbaum [6]. In this sense, PT symmetry is a particular case of antiunitary symmetry where $P \in S_W$, $P : \mathbf{x} \to -\mathbf{x}$. Another particular case of antiunitary symmetry is the partial PT symmetry introduced by Yang [9] which for two-dimensional models takes the form $W_x : (x, y) \to (-x, y)$ or $W_y : (x, y) \to (x, -y)$. As indicated in the introduction Klainman and Cederbaum [6] already chose these coordinate transformations to illustrate the concept of ST symmetry (they called them $P_x$ and $P_y$, respectively). This kind of symmetry was recently extended by Beygi et al [8] to $N$-dimensional oscillators.

If we apply $W \in S_W$ to the eigenvalue equation

$$H(\lambda)\psi_n = E_n(\lambda)\psi_n, \tag{5}$$

we have

$$W H(\lambda) W^{-1} W \psi_n = H(-\lambda) W \psi_n = E_n(\lambda) W \psi_n. \tag{6}$$

It is clear that

$$E_n(\lambda) = E_m(-\lambda), \tag{7}$$

where $E_m(-\lambda)$ is an eigenvalue of $H(-\lambda)$. Since this equation should be valid for all $\lambda$, then when $\lambda \to 0$ we have $E_n(0) = E_m(0)$. If the eigenvalue $E_n(0)$ of $H_0$ is nondegenerate, then $m = n$, $E_n(-\lambda) = E_n(\lambda)$ and perturbation theory yields the formal power series

$$E_n(\lambda) = \sum_{j=0}^{\infty} E_n^{(j)} \lambda^{2j}. \tag{8}$$

Klaiman et al [15] obtained a similar result by means of a lengthier argument. If the radius of convergence of this series is finite, then we conclude that $E_n(\lambda)$ is real for sufficiently small $|\lambda|$ when $\lambda = -\lambda^*$. If, on the other hand, the eigenvalue $E_n(0)$ is degenerate, then the perturbation series for $E_n(\lambda)$ may exhibit odd powers of $\lambda$ and one expects complex eigenvalues under such conditions. This argument is also expected to apply to the more general case of a divergent perturbation series provided it is asymptotic to $E_n(\lambda)$ because we only have to consider sufficiently small values of $|\lambda|$ to prove that the eigenvalue is complex. We appreciate that the occurrence of real eigenvalues of a non-Hermitian operator $H(ig)$, $g$ real, is strongly dependent on the form of the spectrum of $H_0$. This approach based on perturbation theory is not new [3] but we outline it here for completeness.

## 3. Coupled harmonic oscillators

In this section we focus our attention on the set of adjacently coupled harmonic oscillators studied by Beygi et al [8]

$$H(\lambda) = \frac{1}{2} \sum_{i=1}^{N} \left( p_i^2 + \omega_i^2 x_i^2 \right) + \lambda \sum_{j=1}^{N-1} x_j x_{j+1}, \tag{9}$$

where $p_j = -i\partial/\partial x_j$ in the coordinate representation. As indicated in the introduction, the particular case $N = 2$ was studied earlier by Fernández and Garcia [1]. If we choose the oscillator frequencies $\omega_j$ so that the spectrum of $H_0 = H(0)$ is nondegenerate then, arguing as in Sec. 2, the spectrum of $H(ig)$ is expected to be real for sufficiently small values of $g$. When some of the frequencies are equal, degeneracy emerges as well as the chance of complex eigenvalues. Consequently, one expects phase transitions; that

is to say surfaces in the $\omega$-plane that separate regions of real and complex eigenvalues. Fernández and Garcia [1] and more generally and exhaustively Beygi et al [8] discussed some particular cases in detail.

From the point of view of symmetry the case of equal frequencies is the most interesting one; therefore in what follows we choose $\omega_j = 1$ for all $j$. We first obtain the unitary operators $U_j$ that leave $H'$ invariant. They are given by the coordinate transformations

$$\begin{aligned}
&U_1 : (x_1, x_2, \ldots, x_N) \to (x_1, x_2, \ldots, x_N), \\
&U_2 : (x_1, x_2, \ldots, x_N) \to (-x_1, -x_2, \ldots, -x_N), \\
&U_3 : (x_1, x_2, \ldots, x_N) \to (x_N, x_{N-1}, \ldots, x_1), \\
&U_4 : (x_1, x_2, \ldots, x_N) \to (-x_N, -x_{N-1}, \ldots, -x_1).
\end{aligned} \tag{10}$$

The set $G_4 = \{U_1, U_2, U_3, U_4\}$ is an Abelian group isomorphic to $D_2$, $C_{2v}$ and $C_{2h}$ [12,13].

In addition to these four operators there are other four ones that change the sign of $H'$ leaving $H_0$ invariant:

$$\begin{aligned}
&W_1 : \;\; x_j \to (-1)^j x_j, \\
&W_2 : \;\; x_j \to (-1)^{j+1} x_j, \\
&W_3 = U_3 W_1, \\
&W_4 = U_3 W_2.
\end{aligned} \tag{11}$$

We can thus construct four antiunitary operators $A_j = W_j T$, $j = 1, 2, 3, 4$ that leave $H(ig)$ invariant and describe its antiunitary symmetry. Beygi et al [8] only considered $A_1$ and $A_2$ in their discussion of partial PT symmetry. If $A_1$ and $A_2$ describe partial PT symmetry, how do we call the antiunitary symmetries associated to $A_3$ and $A_4$: reverse-order partial PT symmetry? Instead of adding more fancy names to new antiunitary transformations of the Hamiltonian operator we suggest the general term antiunitary symmetry.

The set of eight operators $G_8 = \{U_1, U_2, U_3, U_4, W_1, W_2, W_3, W_4\}$ is also a group, with a structure that depends on $N$. These operators obviously leave $H_0$ invariant

because $G_8$ is a subgroup of the actual symmetry group for $H_0$ that we do not discuss here. $G_8$ is isomorphic to $C_{4v}$ when $N$ is even and to $D_{2h}$ when $N$ is odd [12, 13].

Each eigenvalue $E_k^{(0)}$ of $H_0$ is $g_k$-fold degenerate, where

$$\begin{aligned} E_k^{(0)} &= k + \frac{N}{2},\ k = \sum_{j=1}^{N} n_j,\ n_j = 0, 1, \ldots, \\ g_k &= \frac{(k+N-1)!}{k!(N-1)!}. \end{aligned} \tag{12}$$

Such great degeneracy is the reason why so many eigenvalues of $H(ig)$ are complex. For example, when $N = 2$ all the eigenvalues with $n_1 \neq n_2$ are complex [1, 8]. For any value of $N$ the lowest eigenvalue $E_0^{(0)}$ is nondegenerate and $E_0(ig)$ is real for all $g$ [8] in agreement with the perturbation analysis outlined in Sec. 2.

## 4. Conclusions

Instead of inventing new names for new antiunitary symmetries that may appear in future investigations we suggest to resort to the quite general concept of antiunitary symmetry. It is perfectly reasonable to keep PT symmetry for historical reasons but it is far more convenient to avoid particular names for all its possible variants such as partial PT symmetry, reverse-order partial PT symmetry, etc. and simply use antiunitary symmetry. To this end we have tried to formalize the concept of antiunitary symmetry by means of group theory.

We have also tried to show that perturbation theory is useful for predicting whether a given non-Hermitian Hamiltonian will exhibit antiunitary symmetry, at least for some kind of examples. Although the approach is not completely rigorous it nevertheless provides a fairly good idea of what to expect. As a general rule, the greater the symmetry of $H_0$ the more probable the occurrence of complex eigenvalues for all values of $g$. This point of view was used successfully in earlier papers [1–4].

As an illustrative example of the main theoretical ideas we have chosen the coupled harmonic oscillators studied by Beygi et al [8] and disclosed their point-group symmetry as well as their antiunitary symmetry.

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